for some N. The only difficulty in the proof of these statements, beyond the methods already used in §§1–4, is the possibility that surfaces in $\mathfrak{P}$ may have boundary values which are constant on some arcs of the circumference.

Statement (1) yields a modified Morse theory; the usual requirement is that every k-cap contain a minimal surface $\mathfrak{z}$ for which $D[\mathfrak{z}] = a$. But by assigning type numbers to *blocs* of minimal surfaces, the usual Morse relations hold.

[1] [1] Shiffman, "The Plateau Problem for Non-Relative Minima," *Ann. Math.*, **40**, 834–854 (1940); [2] Morse and Tompkins, "The Existence of Minimal Surfaces of General Critical Type," *Ibid.*, **40**, 443–472 (1940); [3] Courant, "Critical Points and Unstable Minimal Surfaces," these PROCEEDINGS, **27**, 51–57 (1941); [4] Morse and Tompkins, "Minimal Surfaces Not of Minimum Type by a New Mode of Approximation," *Ann. Math.*, **42**, 62–72 (1941); and "The Continuity of the Area of Harmonic Surfaces as a Function of the Boundary Representations," *Am. Jour. Math.*, **63**, 825–838 (1941).

[2] Cf. [3].

[3] Cf. [4].

[4] See Radò, "On the Problem of Plateau," *Ergeb. Math.*, **2**, 45–47 (1933).

[5] The uniformity of the convergence may be eliminated since it is a consequence of $L^n(1) \rightarrow L^\infty(1)$.

[6] Cf. Shiffman, "The Plateau Problem for Minimal Surfaces Which Are Relative Minima," *Ann. Math.*, **39**, 311–312 (1938).

[7] This theorem is capable of very wide generalizations.

[8] In $\mathfrak{P}$ this metric is equivalent to the uniform metric.

[9] $\mathfrak{P}_a$ consists of those surfaces $\mathfrak{x}$ of $\mathfrak{P}$ for which $D[\mathfrak{x}] \leq a$.

ON HOMOGENEOUS MEASURE ALGEBRAS

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1. The purpose of this note is to determine the structure of general measure algebras.

2. For any Boolean σ-algebra M, let $\overline{M}$ be the least cardinal number which is the power of a σ-basis of M. M is *homogeneous*, if $\overline{L} = \overline{M}$ for every (principal) ideal $L \subseteq M$ different from the null ideal. A Boolean σ-algebra $M = \{0 \leqq a, b, c, \ldots \leqq e\}$ is a *measure algebra*, if there is defined on M a *measure function* (that is, a countably additive real non-negative function) $\mu(a)$ such that (i) $0 < \mu(e) < \infty$, (ii) $\mu(a) = 0$ if and only if $a = 0$. We also assume that there is no atomic element in M, i.e., (iii) $a > 0$ implies the existence of a $b \in M$ such that $0 < b < a$

An example of a homogeneous measure algebra is the Boolean algebra $P(\gamma)$ of all measurable sets (mod. null sets) of an infinite product space $\Omega_\gamma = P_{0 \leqq \alpha < \gamma} I_\alpha$ of intervals I_α: $0 \leqq x_\alpha \leqq 1$, where α and γ are ordinal numbers and the measure on Ω_γ is defined multiplicatively in terms of the ordinary Lebesgue measure on each I_α.

If $\gamma < \gamma'$, then $P(\gamma)$ may be considered as a σ-subalgebra of $P(\gamma')$. This embedding is obtained by identifying each subset E of Ω_γ with the cylinder set $E \times P_{\gamma \leqq \alpha < \gamma'} I_\alpha$ in the product space $\Omega_{\gamma'} = \Omega_\gamma \times P_{\gamma \leqq \alpha < \gamma'} I_\alpha$.

Two measure algebras are *isomorphic*, if there exists a measure-preserving σ-isomorphism between them. It is easy to see that, for any infinite ordinals γ and γ', two measure algebras $P(\gamma)$ and $P(\gamma')$ are isomorphic if and only if γ and γ' correspond to the same cardinal. Moreover, if γ is finite or corresponds to $\aleph_0$, then $P(\gamma)$ is isomorphic to $P(1)$, i.e., to the measure algebra of Lebesgue measurable sets (mod. null sets) of the interval I: $0 \leqq x \leqq 1$.

THEOREM 1. *Every homogeneous measure algebra with $\mu(e) = 1$ is isomorphic to $P(\gamma_0)$, where γ_0 is the least ordinal number corresponding to M.*

This theorem will be proved by transfinite induction. It will be sufficient to prove the following

LEMMA 1: *Let L be a σ-subalgebra of a homogeneous measure algebra M with $\mu(e) = 1$ such that $\bar{L} < \bar{M}$, and assume that L is isomorphic to $P(\gamma)$, where γ is an ordinal corresponding to $\bar{L}$. Then, for any $a \in M$ with $a \bar{\in} L$, there exists a σ-subalgebra L' of M such that $L \subset L'$, $a \in L'$, and L' is isomorphic to $P(\gamma + 1)$, in such a way that this isomorphism is an extension of the given isomorphism of L and $P(\gamma)$.*

The proof of this lemma will be given in section 4.

3. Let us assume the conditions of Lemma 1. We can consider Ω_γ as a representation space of $L = P(\gamma)$. Then, by a theorem of Radon-Nikodym, there exists, for any $a \in M$, a measurable function $\varphi_a(\xi)$ defined on Ω_γ such that $0 \leqq \varphi_a(\xi) \leqq 1$, and $\int_{E_x} \varphi_a(\xi) d\xi = \mu(a \wedge x)$ for any $x \in L$, where E_x is a measurable set of Ω_γ which corresponds to $x \in L$. An element $a \in M$ is called *independent* of L, if we have $\varphi_a(\xi) =$ constant [namely, $= \mu(a)$] almost everywhere (a. e.) on Ω_γ. This is equivalent to saying that we have $\mu(a \wedge x) = \mu(a) \cdot \mu(x)$ for all $x \in L$.

LEMMA 2: *For any measurable function $\chi(\xi)$ defined on Ω_γ with $0 \leqq \chi(\xi) \leqq \varphi_a(\xi)$ a. e. on Ω_γ, there exists a $b \in M$ such that $0 \leqq b \leqq a$ and $\varphi_b(\xi) = \chi(\xi)$ a. e. on Ω_γ.*

Proof of Lemma 2.—Our lemma is clear if $\chi(\xi) = 0$ a. e. on Ω_γ. Hence we may assume that $\text{meas}[\xi : \chi(\xi) > 0] > 0$. Because of the principle of exhaustion, it is sufficient to show that there exists a $b^* \in M$ such that $0 < b^* \leqq a$ and $\varphi_{b^*}(\xi) \leqq \chi(\xi)$ a. e. on Ω_γ.

Let A be the principal ideal generated by a and let $L(a)$ be the σ-subalgebra of M generated by L and a. Since $\overline{L(a)} = \bar{L} < \bar{A} = \bar{M}$ by assumption, there exists a $b_1 \epsilon M$ such that $b_1 \epsilon A$ and $b_1 \bar{\epsilon} L(a)$. This means that meas$[\xi: 0 < \varphi_{b_1}(\xi) < \varphi_a(\xi)] > 0$. Again, by the principle of exhaustion, we can further find a $b_2 \epsilon M$ such that $0 < b_2 < a$ and $0 < \varphi_{b_2}(\xi) < \varphi_a(\xi)$ a. e. on the set $[\xi:\varphi_a(\xi) > 0]$. Let us denote by c_1 and c_2 the elements of L which correspond to the sets $[\xi:0 < \varphi_{b_2}(\xi) \leqq 2^{-1}\varphi_a(\xi)]$ and $[\xi:2^{-1}\varphi_a(\xi) < \varphi_{b_2}(\xi) < \varphi_a(\xi)]$. If we now put $b_3 = (c_1 \wedge b_2) \vee (c_2 \wedge (a - b_2))$, then $0 < b_3 < a$ and we have $0 < \varphi_{b_3}(\xi) \leqq 2^{-1}\varphi_a(\xi)$ a. e. on the set $[\xi:\varphi_a(\xi) > 0]$. By iterating this argument, we can find, for each n, a $b_{n+2} \epsilon M$ such that $0 < b_{n+2} < a$ and $0 < \varphi_{b_{n+2}}(\xi) \leqq 2^{-n}\varphi_a(\xi)$ a. e. on the set $[\xi:\varphi_a(\xi) > 0]$. Take n so large that meas $E^* > 0$, where $E^* = [\xi:2^{-n}\varphi_a(\xi) \leqq \chi(\xi)]$, and denote by c^* the element of L which corresponds to E^*. If we put $b^* = c^* \wedge b_{n+2}$, then we have $0 < b^* < a$, and $\varphi_{b^*}(\xi) = \varphi_{b_{n+2}}(\xi) \leqq 2^{-n}\varphi_a(\xi) \leqq \chi(\xi)$ a. e. on E^*, and $\varphi_{b^*}(\xi) = 0$ a. e. on $\Omega_\gamma - E^*$. Hence $\varphi_{b^*}(\xi) \leqq \chi(\xi)$ a. e. on Ω_γ and this proves Lemma 2.

4. *Proof of Lemma 1.* Let Δ_n be the set of all finite sequences $\delta = \{\epsilon_1, \ldots, \epsilon_n\}$, where $\epsilon_i = 0$ or 1, $i = 1,\ldots, n$; and let $\Delta = \bigcup_{n=1}^{\infty}\Delta_n$. A countable set $\{a_\delta\}$ $(\delta \epsilon \Delta)$ is a *dyadic decomposition* of a if $a_0 \vee a_1 = a$, $a_0 \wedge a_1 = 0$, and $a_{\delta,0} \vee a_{\delta,1} = a_\delta, a_{\delta,0} \wedge a_{\delta,1} = 0$ for all $\delta \epsilon \Delta$. By Lemma 2, there exists for any $a \epsilon M$, a dyadic decomposition $\{a_\delta\}(\delta \epsilon \Delta)$ of a such that $\varphi_{a_\delta}(\xi) = \min.(\varphi_a(\xi), \epsilon_1/2 + \ldots + \epsilon_n/2^n + 1/2^n) - \min.(\varphi_a(\xi), \epsilon_1/2 + \ldots + \epsilon_n/2^n)$ a. e. on Ω_γ for all $\delta = \{\epsilon_1, \ldots, \epsilon_n\}\epsilon\Delta$. In the same way, there exists a dyadic decomposition $\{a'_\delta\}(\delta \epsilon \Delta)$ of $a' = e - a$ such that $\varphi_{a'_\delta}(\xi) = \max.(\varphi_a(\xi), \epsilon_1/2 + \ldots + \epsilon_n/2^n + 1/2^n) - \max.(\varphi_a(\xi), \epsilon_1/2 + \ldots + \epsilon_n/2^n)$ a. e. on Ω_δ for all $\delta = \{\epsilon_1, \ldots, \epsilon_n\}\epsilon\Delta$. Let us put $b_\delta = a_\delta \vee a'_\delta$ for all $\delta \epsilon \Delta$. Then $\{b_\delta\}(\delta \epsilon \Delta)$ is a dyadic decomposition of the unit element e, and each b_δ is independent of L, since we have clearly $\varphi_{b_\delta}(\xi) = \varphi_{a_\delta}(\xi) + \varphi_{a'_\delta}(\xi) \equiv 1/2^n$ a. e. on Ω for all $\delta \epsilon \Delta_n$, $n = 1, 2, \ldots$.

Now let L' be the σ-subalgebra of M generated by L and $\{b_\delta\}(\delta\epsilon \Delta)$. L' is obtained by completing the finite algebra L^* consisting of all elements of M of the form: $b^{(n)} = \vee_{\delta \epsilon \Delta_n}(b_\delta \wedge c_\delta)$, where $c_\delta \epsilon L$ for all $\delta \epsilon \Delta_n$, $n = 1, 2\ldots$ It is easy to see that L' is isomorphic to $P(\gamma + 1)$ by an isomorphism which is an extension of the given isomorphism of L and $P(\gamma)$.

Finally, in order to prove that $a \epsilon L'$, consider the set $[\xi:\varphi_a(\xi) \geqq \epsilon_1/2 + \ldots + \epsilon_n/2^n + 1/2^n]$ for each $\delta = \{\epsilon_1,\ldots, \epsilon_n\}\epsilon\Delta$, and let c_δ be the corresponding element of L. If we put $b^{(n)} = \vee_{\delta \epsilon \Delta_n}(b_\delta \wedge c_\delta)$, then we have $b^{(1)} \leqq b^{(2)} \leqq \ldots \leqq b^{(n)} \leqq \ldots \leqq a$ and $\mu(a - b^{(n)}) \leqq {}^1/_{2^n}$ for $n = 1, 2, \ldots$. Hence we have $a = \vee_{n=1}^{\infty} b^{(n)}$ and consequently $a \epsilon L'$. This proves Lemma 1 and so Theorem 1.

5. A measure algebra M is a *direct sum* of (a finite or countably infinite number of) measure algebras M_n, if each M_n is (isomorphic to) a principal ideal of M and if every element $a \epsilon M$ is uniquely expressed in the form:

$a = \vee_{n=1}^{\infty} a_n$, $a_m \wedge a_n = 0 (m \neq n)$, where $a_n \in M_n$, $n = 1, 2, \ldots$. (This last condition is equivalent to saying that the unit elements e_n of M_n, which are elements of M, satisfy $e_m \wedge e_n = 0$ $(m \neq n)$ and $e = \vee_{n=1}^{\infty} e_n$.)

THEOREM 2. *Every measure algebra is a direct sum of homogeneous measure algebras $M_n(n = 1, 2, \ldots$, finite or countably infinite).*

The proof of Theorem 2 is easy and will be omitted.

Theorems 1 and 2 indicate the structure of a general measure algebra.

6. The ergodicity of a measure preserving σ-automorphism T of a measure algebra M (onto itself), or that of a group $G = \{T\}$ of such σ-automorphisms, can be defined as usual.

LEMMA 3. *In order that the group of all measure preserving σ-automorphisms of a measure algebra M be ergodic on M, it is necessary and sufficient that M be homogeneous.*

The proof of this lemma is easy and is omitted.

THEOREM 3. *Let M be a measure algebra, and let G be the group of all measure preserving σ-automorphisms of M. Then M is a direct sum of a countable number of invariant principal ideals M_n, on each of which G is ergodic. This decomposition is the same as in Theorem 2.*

Let G be an arbitrary group of measure preserving σ-automorphisms of M. Then the set L_G of all $a \in M$ such that $T(a) = a$ for all $T \in G$, is a σ-subalgebra of M. Conversely, for any σ-subalgebra $L \subseteq M$, consider the set G_L of all measure preserving σ-automorphisms T of M such that $T(a) = a$ for all $a \in L$. G_L is clearly a group. It is also clear that $L \subseteq L_{G_L}$ and $G \subseteq G_{L_G}$. Exactly as in the theory of Galois, we may ask the question: Under what conditions do the equalities $L = L_{G_L}$ and $G = G_{L_G}$ hold? These and related problems will be discussed elsewhere.

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